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THE WEDDERBURN PRINCIPAL THEOREM FOR A GENERALIZATION OF ALTERNATIVE ALGEBRAS

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ABSTRACT. A generalized alternative ring I is a nonassociative ring R in which the identities $(wx,y,z) + (w,x,[y,z]) - w(x,y,z) - (w,y,z)x$; $([w,x],y,z) + (w,x,yz) - y(w,x,z) - (w,x,y)z$; and (x,x,x) are identically zero. It is here demonstrated that if A is a finite-dimensional algebra of this type over a field F of characteristic $\neq 2, 3$, then A a nilalgebra implies A is nilpotent.

A generalized alternative ring II is a nonassociative ring R in which the identities $(wx,y,z) + (w,x,[y,z]) - w(x,y,z) - (w,y,z)x$ and (x,y,x) are identically zero. Let A be a finite-dimensional algebra of this type over a field F of characteristic $\neq 2$. Then it is here established that (1) A a nilalgebra implies A is nilpotent; (2) A simple with no nonzero idempotent other than 1 and F algebraically closed imply A itself is a field; and (3) the standard Wedderburn principal theorem is valid for A.

1. **Preliminaries.** Let R be a nonassociative ring. As is customary, for $x, y, z \in R$ we denote by (x,y,z) the associator $(x,y,z) = (xy)z - x(yz)$ and by $[x,y]$ the commutator $[x,y] = xy - yx$. A straightforward verification shows that the following identity, known as the Teichmüller identity, holds for all $w, x, y, z \in R$:

$$(wx,y,z) - (w,xy,z) + (w,x,yz) = w(x,y,z) + (w,x,y)z. \tag{T}$$

A nonassociative ring R is called power-associative if for every $x \in R$ the subring generated by x is associative.

A. *Generalized alternative rings* I. In [4] Kleinfeld defines a generalized alternative ring I to be a nonassociative ring R such that for all $w, x, y, z \in R$ the following identities are satisfied:

$$(wx,y,z) + (w,x,[y,z]) = w(x,y,z) + (w,y,z)x, \tag{1.1}$$

$$([w,x],y,z) + (w,x,yz) = y(w,x,z) + (w,x,y)z, \tag{1.2}$$

$$(x,x,x) = 0. \tag{1.3}$$

That such a ring is power-associative can be readily verified as follows:

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Theorem 1.1. *A generalized alternative ring* I *is power-associative.*

Proof. Define $x^n = x^{n-1}x$. We need to show $x^i x^j = x^{i+j}$ for any $i, j > 0$. From (1.3) we have $x^3 = x^2 x = x x^2$. Also, (1.1) and (1.3) yield $(x^2, x, x) + (x, x, [x, x]) = x(x, x, x) + (x, x, x)x$ or $(x^2, x, x) = 0$, which implies $x^4 = x^3 x = x^2 x^2$; while (1.2) and (1.3) yield $([x, x], x, x) + (x, x, x^2) = x(x, x, x) + (x, x, x)x$ or $(x, x, x^2) = 0$, which implies $x^2 x^2 = x x^3$.

The proof is now by induction. We assume $x^i x^j = x^{i+j}$ for $i + j < n$; $i, j > 0$ and $n > 4$. Then (1.1) gives $(x^2, x^{n-2-i}, x^i) + (x, x, [x^{n-2-i}, x^i]) = x(x, x^{n-2-i}, x^i) + (x, x^{n-2-i}, x^i)x$ or $(x^2, x^{n-2-i}, x^i) = 0$, using the induction assumption. Thus, except possibly for $i = n - 1$, $x^{n-i}x^i = x^2 x^{n-2}$. But, again using the induction assumption, (1.1) gives $(x^2, x^{n-3}, x) + (x, x, [x^{n-3}, x]) = x(x, x^{n-3}, x) + (x, x^{n-3}, x)x$ or $(x^2, x^{n-3}, x) = 0$, that is $x^n = x^{n-1}x = x^2 x^{n-2}$. Thus $x^{n-i}x^i = x^n$, except possibly for $i = n - 1$. Finally, (1.2) and the induction assumption yield $([x, x^{n-3}], x, x) + (x, x^{n-3}, x^2) = x(x, x^{n-3}, x) + (x, x^{n-3}, x)x$ or $(x, x^{n-3}, x^2) = 0$, that is $x x^{n-1} = x^{n-2}x^2 = x^n$. This completes the induction.

Let R be a generalized alternative ring I. If one defines a new multiplication for R by $x * y = yx$, then a straightforward verification shows that under this new multiplication identity (1.1) is converted to identity (1.2) and vice versa. Thus, since identity (1.3) is left unchanged, the resulting ring is itself a generalized alternative ring I. We henceforth refer to this procedure as passing to the anti-isomorphic copy of R.

In this work we consider generalized alternative algebras I over fields of characteristic $\neq 2, 3$. In addition to the above defining identities, we also make repeated use of the following:

$$(y, x, x) + (x, y, x) + (x, x, y) = 0, \tag{1.4}$$

$$(w, xy, z) - (w, x, zy) + (w, x, y)z - (w, y, z)x = 0, \tag{1.5}$$

$$(z, yx, w) - (yz, x, w) + z(y, x, w) - x(z, y, w) = 0, \tag{1.6}$$

$$(x, x, yx) = (x, x, y)x, \tag{1.7}$$

$$(x, xy, x) = x(x, y, x), \tag{1.8}$$

$$(x^2, y, x) = 2x(x, y, x), \tag{1.9}$$

$$(x^2, x, y) = (x, x^2, y) = 2(x, x, yx). \tag{1.10}$$

Identity (1.4) is obtained from linearization of (1.3). Identity (1.5) is obtained by subtracting (T) from (1.1), after which (1.6) follows from (1.5) by passing to the anti-isomorphic copy of R. Identities (1.7), (1.8), and (1.9) are established in [4]. To see that $(x^2, x, w) = (x, x^2, w)$, we let $z = y = x$ in (1.6); while to see that $(x, x^2, z) = 2(x, x, zx)$, we let $w = y = x$ in (1.5) and then apply (1.3) and (1.7). Now taking w, z to be y, these last two equations together give (1.10).

B. *Generalized alternative rings* II. A generalized alternative ring II is defined by Kleinfeld in [5] to be a nonassociative ring R such that for all $w, x, y, z \in R$ the following identities are satisfied:

$$(wx,y,z) + (w,x,[y,z]) = w(x,y,z) + (w,y,z)x, \tag{2.1}$$

$$(x,y,x) = 0. \tag{2.2}$$

From these identities one easily generates:

$$(x,y,z) = -(z,y,x), \tag{2.3}$$

$$(w,xy,z) - (w,x,zy) + (w,x,y)z - (w,y,z)x = 0, \tag{2.4}$$

$$(wx,z,z) = w(x,z,z) + (w,z,z)x, \tag{2.5}$$

$$(z,x,z^2) = 0, \tag{2.6}$$

$$z(z,z,x) = (z,z,zx) = (z,z,xz) = (z,z,x)z. \tag{2.7}$$

Identity (2.3) follows from linearization of (2.2). Identity (2.4) is obtained by subtracting (T) from (2.1). If one takes $y = z$ in (2.1), one obtains (2.5). Letting $w = y = z$ in (2.4) and applying (2.2), one obtains (2.6).

To see that $z(z,z,x) = (z,z,zx)$, we let $w = y = z$ in (2.1) and then apply (2.2) and (2.3). To see that $(z,z,zx) = (z,z,x)z$, we take w, x, z to be z and y to be x in (2.4) and then apply (2.2). Finally, to see that $(z,z,x)z = (z,z,xz)$, we take x, y, z to be z and w to be x in (2.1) and then apply (2.2) and (2.3). This establishes (2.7).

A nonassociative ring which satisfies identities (2.2) and (2.6) is called noncommutative Jordan. From [10] an algebra of this type over a field of characteristic $\neq 2$ is known to be power-associative.

2. **Finite-dimensional nilalgebras.** Let A be a power-associative algebra. An element $x \in A$ is said to be nilpotent if there exists an integer $k > 0$ for which $x^k = 0$. Should every element of the algebra A be nilpotent, then A is called a nilalgebra. For any algebra A one obtains a derived series of subalgebras $A^{(0)} \supseteq A^{(1)} \supseteq \cdots$ by defining inductively $A^{(0)} = A$, $A^{(i+1)} = (A^{(i)})^2$. A is called solvable in case $A^{(m)} = 0$ for some integer $m > 0$. A nonassociative algebra A is called nilpotent in case there exists an integer $n > 0$ such that any product $x_1 x_2 \cdots x_n$ of n elements $x_i \in A$, no matter how associated, is zero.

For an algebra A and $x \in A$, the linear operators on A of right and left multiplication by x are denoted by R_x and L_x, respectively. Let $M(A)$ denote the subalgebra generated by all right and left multiplications of A in the associative algebra of all linear operators on A. If B is any subset of A, we shall write B^* for the subalgebra of $M(A)$ generated by all right and left multiplications of A which correspond to elements of B.

A. *Generalized alternative algebras* I.

Lemma 1.1. *Let B be a generalized alternative algebra* I *over a field F of characteristic $\neq 2, 3$. Suppose $B = Fb + C$ where C is a subalgebra of B such that $B^2 \subsetneq C$. If $H = B^* C^* + C^*$, then $S_x S_y S_z \in H$ for $S = L$ or R and all $x, y, z \in B$.*

Proof. We begin by making some reductions. First, since every product of three operators of right or left multiplication corresponding to elements of $B = Fb + C$ may be expressed as a linear combination of products $S_x S_y S_z$ where each of x, y, z is either in C or equal to b, it suffices to verify only that products of this latter form belong to H. In particular, since $c \in C$ clearly implies $S_x S_y S_c \in H$, we need only consider products of the form $S_x S_y S_b$ where each of x and y is either in C or equal to b.

Secondly, should $S_{x_1} S_{x_2} S_{x_3} \in H$, then by passing to the anti-isomorphic copy of B one sees that $S'_{x_1} S'_{x_2} S'_{x_3} \in H$, where $S'_{x_i} = R_{x_i}(L_{x_i})$ if $S_{x_i} = L_{x_i}(R_{x_i})$ for $i = 1, 2, 3$.

Henceforth let $c, c' \in C$. From (1.1), $(wc,y,b) + (w,c,[y,b]) = w(c,y,b) + (w,y,b)c$, we have $R_c R_y R_b - R_c R_{by} + R_{c(by)} = R_{(cy)b} + R_y R_b R_c - R_{yb} R_c$. Since by assumption $B^2 \subsetneq C$, this yields $R_c R_y R_b \in H$ or

(1.a) $R_c R_{c'} R_b,\ R_c R_b R_b \in H$.

Now applying our second reduction, (1.a) in turn yields

(1.b) $L_c L_{c'} L_b,\ L_c L_b L_b \in H$.

From (1.2), $([b,x],y,c) + (b,x,yc) = y(b,x,c) + (b,x,y)c$, we have $-L_{xb} R_c + R_c L_{xb} - R_c L_x L_b = R_{(b,x,c)} - L_x L_b R_c$. Again using the assumption $B^2 \subsetneq C$, as we will continually do throughout, this yields $R_c L_x L_b \in H$ or

(1.c) $R_c L_{c'} L_b,\ R_c L_b L_b \in H$.

Again applying our second reduction, as we also will continually do throughout, (1.c) in turn yields

(1.d) $L_c R_{c'} R_b,\ L_c R_b R_b \in H$.

From (1.1), $(c'b,y,c) + (c',b,[y,c]) = c'(b,y,c) + (c',y,c)b$, using (1.d) we now obtain

(1.e) $R_c L_{c'} R_b,\ L_c R_{c'} L_b \in H$.

Adding (1.1), $(bx,c,c') + (b,x,[c,c']) = b(x,c,c') + (b,c,c')x$, to (1.2), $([b,x], c',c) + (b,x,c'c) = c'(b,x,c) + (b,x,c')c$, we have

(1.f) $R_c R_{c'} L_b,\ L_c L_{c'} R_b \in H$.

Linearization of (1.10), $(b^2,b,y) = 2(b,b,yb)$, gives

$$(b^2,c,y) + (bc,b,y) + (cb,b,y) = 2[(b,b,yc) + (b,c,yb) + (c,b,yb)].$$

Using (1.c) this yields $2R_b L_c L_b \in H$ or

(1.g) $R_b L_c L_b,\ L_b R_c R_b \in H$.

From (1.6) we obtain $(z,bc,b) - (bz,c,b) + z(b,c,b) - c(z,b,b) = 0$. Using (1.g) this implies $R_{bc} R_b \in H$. Then from (1.1), $(wb,b,c) + (w,b,[b,c]) = w(b,b,c) + (w,b,c)b$, we have

(1.h) $R_bR_cR_b$, $L_bL_cL_b \in H$.

Next from (1.5), $(b,cy,b) - (b,c,by) + (b,c,y)b - (b,y,b)c = 0$, if we use (1.h) we obtain $L_{bc}R_b - L_cR_bL_b \in H$. Since from (1.6), $(c,bx,b) - (bc,x,b) + c(b,x,b) - x(c,b,b) = 0$, one has $L_bL_cR_b - L_{bc}R_b \in H$, adding this to $L_{bc}R_b - L_cR_bL_b$ gives

(i.1) $L_bL_cR_b - L_cR_bL_b \in H$.

From (1.4), $(cx,b,b) + (b,cx,b) + (b,b,cx) = 0$, using (1.b) and (1.d) one also has

(i.2) $L_cR_bL_b - L_cL_bR_b \in H$.

If we now linearize (1.8) to obtain $(c,by,b) + (b,cy,b) + (b,by,c) = c(b,y,b) + b(c,y,b) + b(b,y,c)$, then using (1.c), (1.g), (i.1), and (i.2) we have $2L_bR_cL_b \in H$ or

(1.i) $L_bR_cL_b$, $R_bL_cR_b \in H$.

From linearization of (1.4) we obtain $(bx,b,c) + (b,c,bx) + (c,bx,b) + (bx,c,b) + (c,b,bx) + (b,bx,c) = 0$. Using (1.g), (1.h), and (1.i) this yields

(1.j) $L_bL_cR_b$, $R_bR_cL_b \in H$.

From (1.1), $(b^2,y,c) + (b,b,[y,c]) = b(b,y,c) + (b,y,c)b$, using (1.b), (1.g), and (1.i) we have

(1.k) $R_cL_bR_b$, $L_cR_bL_b \in H$.

Next (1.4), $(xc,b,b) + (b,xc,b) + (b,b,xc) = 0$, together with (1.a), (1.c), and (1.k) gives

(1.m) $R_cR_bL_b$, $L_cL_bR_b \in H$.

From (1.10), $(b^2,b,y) = 2(b,b,yb)$, we obtain $2R_bL_bL_b \in H$, that is

(1.n) $R_bL_bL_b$, $L_bR_bR_b \in H$.

We next add (1.1), $(b^2,x,b) + (b,b,[x,b]) = b(b,x,b) + (b,x,b)b$, to (1.2), $([b,x],b,b) + (b,x,b^2) = b(b,x,b) + (b,x,b)b$ to derive $(b^2,x,b) + (b,x,b^2) + (b,b,[x,b]) + ([b,x],b,b) = 2b(b,x,b) + 2(b,x,b)b$. Since (1.9) gives $2b(b,x,b) = (b^2,x,b)$, and since by passing to the anti-isomorphic copy of B this in turn gives $2(b,x,b)b = (b,x,b^2)$, our last equation simplifies to $(b,b,[x,b]) + ([b,x],b,b) = 0$. Then using (1.6), $(xb,b,b) = (b,xb,b) + b(x,b,b) - b(b,x,b)$, we have $(b,b,[x,b]) + (bx,b,b) = (b,xb,b) + b(x,b,b) - b(b,x,b)$. Now by passing to the anti-isomorphic copy of B, (1.8) and (1.7) become $(b,xb,b) = (b,x,b)b$ and $b(x,b,b) = (bx,b,b)$, respectively. Hence our equation again simplifies, this time to $(b,b,[x,b]) = [(b,x,b),b]$. Using (1.n) this gives

(p.1) $L_bL_bL_b + R_bL_bR_b + L_bR_bL_b \in H$.

From (1.4), $(bx,b,b) + (b,b,bx) + (b,bx,b) = 0$, using (1.8) we obtain $(bx,b,b) = -(b,b,bx) - b(b,x,b)$. If we use (1.n), this yields

(p.2) $L_bL_bL_b - L_bR_bL_b \in H$.

Now subtracting (p.2) from (p.1) we have

(p.3) $R_bL_bR_b + 2L_bR_bL_b \in H$.

Next from linearization of (1.4) we obtain $(x,b,b^2) + (b,b^2,x) + (b^2,x,b) + (x,b^2,b) + (b^2,b,x) + (b,x,b^2) = 0$ or $2(x,b,b^2) + 2(b^2,b,x) + (b^2,x,b) + (b,x,b^2) = 0$, since (1.10) implies $(b,b^2,x) = (b^2,b,x)$, and since by passing to the anti-isomorphic copy of B this in turn implies $(x, b^2, b) = (x, b, b^2)$. Thus we have

(p.4) $(L_{b^2})R_b - (R_{b^2})L_b \in H$.

Now (1.9) gives $(b^2, x, b) = 2b(b, x, b)$, and by passing to the anti-isomorphic copy of B this in turn gives $(b, x, b^2) = 2(b, x, b)b$. Hence using (1.n) we also have

(p.5) $2L_b R_b L_b - (L_{b^2})R_b \in H$, and

(p.6) $2R_b L_b R_b - (R_{b^2})L_b \in H$.

If we now subtract (p.4) and (p.5) from (p.6), we obtain

(p.7) $2R_b L_b R_b - 2L_b R_b L_b \in H$.

Lastly, adding (p.7) to (p.3) we have $3R_b L_b R_b \in H$ or

(1.p) $R_b L_b R_b,\ L_b R_b L_b \in H$.

In addition, (p.2) and (1.p) together also show

(1.q) $L_b L_b L_b,\ R_b R_b R_b \in H$.

Finally, by passing to the anti-isomorphic copy of B, from (1.8) we obtain as before $(b, xb, b) = (b, x, b)b$. Using (1.n) and (1.p) we then have

(1.r) $R_b R_b L_b,\ L_b L_b R_b \in H$.

This completes the proof of the lemma.

From Schafer's proof of Theorem 3 in [15], which proof in turn is modelled on that of Albert for standard algebras in [2], it follows that Lemma 1.1 is sufficient to obtain the following result.

Theorem 1.2. *Let A be a finite-dimensional generalized alternative algebra* I *over a field F of characteristic $\neq 2, 3$. If B is a solvable subalgebra of A, then B^* is nilpotent.*

Lemma 1.2. *Let A be a generalized alternative algebra* I *over a field F of characteristic $\neq 2, 3$; and let B be a subalgebra of A. If $x \in A$ is such that $xB \subseteq B$, $Bx \subseteq B$, then $(x^2B)B \subseteq B$, $x^2B^2 \subseteq B$, $B(x^2B) \subseteq B$, $B^2x^2 \subseteq B$, and $(x^2B)^2B \subseteq B$.*

Proof. We assume throughout that $b_i \in B$ for $i = 1, 2, 3$. From (1.5) we have $(x, b_1x, b_2) - (x, b_1, b_2x) + (x, b_1, x)b_2 - (x, x, b_2)b_1 = 0$ or $(x^2B)B \subseteq B$. Then from (1.5) we also have $(x, xb_1, b_2) - (x, x, b_2b_1) + (x, x, b_1)b_2 - (x, b_1, b_2)x = 0$ or $x^2B^2 \subseteq B$. Next (1.6) gives $(x, xb_2, b_1) - (x^2, b_2, b_1) + x(x, b_2, b_1) - b_2(x, x, b_1) = 0$ or $B(x^2B) \subseteq B$. Now (1.6) also gives $(b_1, xb_2, x) - (xb_1, b_2, x) + b_1(x, b_2, x) - b_2(b_1, x, x) = 0$ or $B(Bx^2) \subseteq B$, whence (1.6) yields $(b_1, b_2x, x) - (b_2b_1, x, x) + b_1(b_2, x, x) - x(b_1, b_2, x) = 0$ or $B^2x^2 \subseteq B$.

There remains only to show $(x^2B)^2B \subseteq B$. We first observe that from (1.5) we have $(b_1, b_2x, x) - (b_1, b_2, x^2) + (b_1, b_2, x)x - (b_1, x, x)b_2 = 0$ or $(Bx^2)B \subseteq B$. Also, (1.6) then implies $(x^2, b_1(b_2x^2), b_3) - (b_1x^2, b_2x^2, b_3) + x^2(b_1, b_2x^2, b_3) - (b_2x^2)(x^2, b_1, b_3) = 0$ or $(Bx^2)^2B \subseteq B$. Now using (1.4), (1.1) gives

$$[(x^2b_1)b_2]x^2 - (x^2b_1)(x^2b_2) + x^2[b_1(x^2b_2)] - x^2[(b_1b_2)x^2]$$
$$= (x^2, b_2, x^2)b_1 = -(b_2, x^2, x^2)b_1 - (x^2, x^2, b_2)b_1;$$

and (1.2) gives

$$-[(b_1x^2)b_2]x^2 + (b_1x^2)(b_2x^2) - x^2[b_1(b_2x^2)] + [x^2(b_1b_2)]x^2$$
$$= b_2(x^2,b_1,x^2) = -b_2(x^2,x^2,b_1) - b_2(b_1,x^2,x^2).$$

Adding these last two equations and using (1.1), (1.2), and (1.4), we have

$$[(x^2b_1)b_2]x^2 - (x^2b_1)(x^2b_2) + x^2[b_1(x^2b_2)] - x^2[(b_1b_2)x^2]$$
$$- [(b_1x^2)b_2]x^2 + (b_1x^2)(b_2x^2) - x^2[b_1(b_2x^2)] + [x^2(b_1b_2)]x^2$$
$$= -b_2(b_1,x^2,x^2) - (b_2,x^2,x^2)b_1 - b_2(x^2,x^2,b_1) - (x^2,x^2,b_2)b_1$$
$$= -(b_2b_1,x^2,x^2) - (x^2,x^2,b_2b_1)$$
$$= (x^2,b_2b_1,x^2).$$

Finally, multiplication of this last equation on the right by b_3 yields $(x^2B)^2B \subseteq B$.

Lemma 1.3. *Let A be a generalized alternative algebra* I *over a field F of characteristic $\neq 2, 3$; and let B be a subspace of A. If $x \in A$ is such that $xB \subseteq B$, $Bx \subseteq B$, $x^2B \subseteq B$, then $x^kB \subseteq B$, $Bx^k \subseteq B$ for $k = 1, 2, 3, \ldots$.*

Proof. Let $b \in B$. We note that (1.4), $(b,x,x) + (x,b,x) + (x,x,b) = 0$, implies $Bx^2 \subseteq B$. Hence we have $x^kB \subseteq B$, $Bx^k \subseteq B$ for $k = 1, 2$. The proof now is by induction. We assume $x^kB \subseteq B$, $Bx^k \subseteq B$ for $k < n$, $n > 2$. From (1.10) and (1.7) we have $(x,x^2,y) = 2(x,x,y)x$. Linearization of this identity gives

$$(x^{n-2},x^2,b) + (x,x^{n-2}x,b) + (x,xx^{n-2},b)$$
$$= 2[(x^{n-2},x,b)x + (x,x^{n-2},b)x + (x,x,b)x^{n-2}].$$

Applying the induction assumption, we now have $3x^nb \in B$ or $x^nB \subseteq B$. Next linearization of (1.4) gives $(b,x,x^{n-1}) + (b,x^{n-1},x) + (x,b,x^{n-1}) + (x^{n-1},b,x) + (x,x^{n-1},b) + (x^{n-1},x,b) = 0$. Again applying the induction assumption, we have $2bx^n \in B$ or $Bx^n \subseteq B$, and our induction is complete.

The proof of the following theorem is now the same as that of Theorem 4 in [15], with the one exception that, since a generalized alternative algebra I is not necessarily noncommutative Jordan, we need to make use of our Lemma 1.3 in addition to Theorem 1.2 and Lemma 1.2 above.

Theorem 1.3. *Let A be a finite-dimensional generalized alternative algebra* I *over a field F of characteristic $\neq 2, 3$. If A is a nilalgebra, then A is nilpotent.*

B. *Generalized alternative algebras* II.

Lemma 2.1. *Let B be a generalized alternative algebra* II *over a field F of characteristic $\neq 2$. Suppose $B = Fb + C$ where C is a subalgebra of B such that*

$B^2 \subseteq C$. If $H = B^*C^* + C^*$, then $S_xS_yS_z \in H$ for $S = L$ or R and all $x, y, z \in B$.

Proof. As in the proof of Lemma 1.1, it suffices to verify only that H contains products of the form $S_xS_yS_b$ where each of x and y is either in C or equal to b. In addition, we note that (2.2) implies $L_bR_b = R_bL_b$.

Throughout we assume c, $c' \in C$. From (2.1), $(wc,y,b) + (w,c,[y,b]) = w(c,y,b) + (w,y,b)c$, we have $R_cR_yR_b - R_cR_{by} + R_{c(by)} = R_{(cy)b} + R_yR_bR_c - R_{yb}R_c$. Since by assumption $B^2 \subseteq C$, this last equation yields $R_cR_yR_b \in H$ or

(2.a) $R_cR_{c'}R_b, R_cR_bR_b \in H$.

From (2.1), $(cx,y,b) + (c,x,[y,b]) = c(x,y,b) + (c,y,b)x$, we also obtain $L_cR_yR_b - L_cR_{by} + R_{by}L_c = R_yR_bL_c + L_{(c,y,b)}$. Again using the assumption $B^2 \subseteq C$, as we will continually do throughout, this yields $L_cR_yR_b \in H$ or

(2.b) $L_cR_{c'}R_b$, $L_cR_bR_b \in H$.

Now (2.3), $(xc,y,b) = -(b,y,xc)$, gives $R_cR_yR_b - R_cR_{yb} = -R_cL_{by} + R_cL_yL_b$. Using (2.a) this implies $R_cL_yL_b \in H$ or

(2.c) $R_cL_{c'}L_b$, $R_cL_bL_b \in H$.

From (2.3), $(cw,y,b) = -(b,y,cw)$, we also have $L_cR_yR_b - L_cR_{yb} = -L_cL_{by} + L_cL_yL_b$. Using (2.b) this gives $L_cL_yL_b \in H$ or

(2.d) $L_cL_{c'}L_b$, $L_cL_bL_b \in H$.

If we take $w = x = y$ in (2.4) and apply (2.2), we obtain

$$(2.8) \qquad (y,y^2,z) = (y,y,zy) + (y,y,z)y.$$

Taking $x = y = z$ in (2.4) and applying (2.3), we also have

$$(2.9) \qquad (z^2,z,w) = (z,z^2,w).$$

Now (2.8), (2.9), (2.7), and (2.3) together imply

$$(2.10) \qquad (b^2,b,x) = 2(b,b,bx) = -2(xb,b,b),$$

whence $2L_bL_bL_b$, $2R_bR_bR_b \in H$ or

(2.e) $L_bL_bL_b$, $R_bR_bR_b \in H$.

From (2.7), $(b,b,bx) = (b,b,x)b$, we have $L_bL_{b^2} - L_bL_bL_b = (L_{b^2})R_b - L_bL_bR_b$. Since (2.6) and (2.3) imply $(b^2,x,b) = 0$ or $(L_{b^2})R_b = R_b(L_{b^2})$, this last equation is equivalent to $L_bL_{b^2} - L_bL_bL_b = R_bL_{b^2} - L_bL_bR_b$. Using (2.e) we now have

(2.f) $L_bL_bR_b = L_bR_bL_b = R_bL_bL_b \in H$.

From (2.7) and (2.3) we next obtain $(xb, b, b) = (bx,b,b)$. If we again use (2.e), this gives

(2.g) $L_bR_bR_b = R_bL_bR_b = R_bR_bL_b \in H$.

From (2.1) and (2.3) we have $-(c,x,c'b) + (c',b,[x,c]) = c'(b,x,c) + (c',x,c)b$. Using (2.b) this yields

(2.h) $R_cL_{c'}R_b \in H$.

Then from (2.3), $(b,xc,c') = -(c',xc,b)$, using (2.h) we obtain

(2.i) $R_c R_{c'} L_b \in H$.

From (2.1) and (2.3) we also have $-(c,x,bc') + (b,c',[x,c]) = b(c',x,c) + (b,x,c)c'$. Using (2.d) this yields

(2.j) $L_{c'} R_c L_b \in H$.

Then from (2.3), $(b,cx,c') = -(c',cx,b)$, using (2.j) we obtain

(2.k) $L_c L_{c'} R_b \in H$.

Linearization of (2.10) gives

$$(b^2,c,x) + (bc,b,x) + (cb,b,x) = 2[(b,b,cx) + (b,c,bx) + (c,b,bx)]$$
$$= -2[(xb,b,c) + (xb,c,b) + (xc,b,b)].$$

If we now use (2.a) and (2.d), we have $2L_b L_c L_b$, $2R_b R_c R_b \in H$ or

(2.m) $L_b L_c L_b$, $R_b R_c R_b \in H$.

Then from (2.3), $(b,c,bx) = -(bx,c,b)$ and $(b,c,xb) = -(xb,c,b)$, using (2.m) we obtain

(2.n) $L_b R_c R_b$, $R_b L_c L_b \in H$.

Next (2.4), (2.2), and (2.3) yield $(bc,x,b) = -(b,x,c)b$; while (2.1), (2.2), and (2.3) yield $(bc,x,b) + (b,c,[x,b]) = -b(b,x,c)$. Subtracting the first of these equations from the second, we have $(b,c,[x,b]) = -b(b,x,c) + (b,x,c)b$. Also, from (2.1) one has $(b^2,x,c) + (b,b,[x,c]) = b(b,x,c) + (b,x,c)b$. Adding this equation to the one just prior, we obtain $(b^2,x,c) + (b,b,[x,c]) + (b,c,[x,b]) = 2(b,x,c)b$. Using (2.c), (2.d), (2.m), and (2.n) this now gives $2R_c L_b R_b \in H$ or

(2.p) $R_c L_b R_b = R_c R_b L_b \in H$.

From (2.4) and (2.3) we have $(c,xb,b) + (b^2,x,c) - (b,x,c)b + (b,b,c)x = 0$. If we use (2.n) and (2.p), this yields

(2.q) $R_b L_c R_b \in H$.

Then (2.3), $(b,xb,c) = -(c,xb,b)$, using (2.q) gives

(2.r) $R_b R_c L_b \in H$.

From (2.4) and (2.3) we also have $(b,bx,c) + (cx,b,b) - (x,b,b)c + (c,x,b)b = 0$. If we use (2.b) and (2.q), this yields

(2.s) $L_b R_c L_b \in H$.

Then (2.3), $(b,bx,c) = -(c,bx,b)$, using (2.s) gives

(2.t) $L_b L_c R_b \in H$.

From (2.4) and (2.3) we next obtain $(x,b^2,c) - (xb,b,c) + (b,b,c)x - (x,b,c)b = 0$. Using (2.m) this gives

(u.1) $R_{bc} R_b \in H$.

Again using (2.4) and (2.3) we have $(b,xc,b) - (bc,x,b) + (c,x,b)b - (b,c,b)x = 0$, that is $(bc,x,b) = (c,x,b)b$ using (2.2). If we use (2.b) and (2.q), this gives

(u.2) $L_{bc} R_b \in H$.

Lastly, (2.4) and (2.3) imply $(x,bc,b) + (b,b,xc) - (b,b,c)x + (b,c,x)b = 0$. Using (2.c), (u.1), and (u.2) we have

(2.u) $L_c L_b R_b = L_c R_b L_b \in H$.

This completes the proof of the lemma.

As in the case of generalized alternative algebras I, using Lemma 2.1 the following result now follows from the proof of Theorem 3 in [15].

Theorem 2.1. *Let A be a finite-dimensional generalized alternative algebra* II *over a field F of characteristic $\neq 2$. If B is a solvable subalgebra of A, then B^* is nilpotent.*

Corollary. *Let A be a generalized alternative algebra* II *over a field F of characteristic $\neq 2$. If x is a nilpotent element of A, then R_x is nilpotent.*

Lemma 2.2. *Let A be a generalized alternative algebra* II *over a field F of characteristic $\neq 2$, and let B be a subalgebra of A. If $x \in A$ is such that $xB \subseteq B$, $Bx \subseteq B$, then $(x^2B)B \subseteq B$, $x^2B^2 \subseteq B$, $B^2x^2 \subseteq B$, $B(x^2B) \subseteq B$, and $(x^2B)^2B \subseteq B$.*

Proof. We assume throughout that $b_i \in B$ for $i = 1, 2, 3$. First using (2.4) and (2.2) we have $(x, b_1x, b_2) - (x, b_1, b_2x) - (x, x, b_2)b_1 = 0$ or $(x^2B)B \subseteq B$. Now from (2.4) we obtain $(x, xb_1, b_2) - (x, x, b_2b_1) + (x, x, b_1)b_2 - (x, b_1, b_2)x = 0$ or $x^2B^2 \subseteq B$. Then (2.3), $(b_1b_2, x, x) = -(x, x, b_1b_2)$, gives $B^2x^2 \subseteq B$. Next from (2.5) and (2.3) we have $(b_1b_2, x, x) = -b_1(x, x, b_2) - (x, x, b_1)b_2$ or $B(x^2B) \subseteq B$. Finally, (2.4), (2.2), and (2.3) give $(x, b_1x, b_2) - (x, b_1, b_2x) + (b_2, x, x)b_1 = 0$ or $(Bx^2)B \subseteq B$. Since (2.4) and (2.2) yield $(x^2, b_1, x^2b_2) = (x^2, b_1, b_2)x^2$, this in turn gives $(x^2, b_1, x^2b_2)b_3 = [(x^2, b_1, b_2)x^2]b_3$ in $(Bx^2)B \subseteq B$. But then $[x^2[b_1(x^2b_2)]]b_3$ in $(x^2B)B \subseteq B$ implies $[(x^2b_1)(x^2b_2)]b_3 \in B$, that is $(x^2B)^2B \subseteq B$.

Using Theorem 2.1 and Lemma 2.2, the proof of the following theorem is now the same as the proof of Theorem 4 in [15].

Theorem 2.2. *Let A be a finite-dimensional generalized alternative algebra* II *over a field F of characteristic $\neq 2$. If A is a nilalgebra, then A is nilpotent.*

Theorem 2.3. *Let A be a simple, finite-dimensional, generalized alternative algebra* II *over an algebraically closed field F of characteristic $\neq 2$. If A has no nonzero idempotent other than* 1, *then A is itself a field.*

Proof. Since A a simple algebra implies $A^2 = A$, A cannot be nilpotent. Thus the finite-dimensionality of A and Theorem 2.2 imply that A is not a nilalgebra. Proposition 3.3 on p. 32 of [13] then ensures the existence in A of a nonzero idempotent, which by assumption must be 1. Now if characteristic $F = 0$, from [6] it is known that A is itself a field. On the other hand, if characteristic $F \neq 0$ and A is not a field, then A is a nodal algebra, that is $A = F1 + N$ where N consists of nilpotent elements but is not a subalgebra of A. Now since from our earlier corollary we know that x nilpotent implies R_x nilpotent, it follows from Lemma 3 of [12] that A cannot be nodal. Hence A must be a field.

3. The Wedderburn principal theorem. Let A be a power-associative algebra over a field F of characteristic $\neq 2$ and define $x \circ y = \frac{1}{2}(xy + yx)$ for $x, y \in A$. If A contains an idempotent e, then Albert has shown in [2] that $A = A_1 + A_{1/2}$

$+ A_0$ where $A_i = \{x \in A\colon x \circ e = ix\}$. In fact, $ex = x = xe$ for $x \in A_1$ and $ex = 0 = xe$ for $x \in A_0$. This decomposition of A is known as the Albert decomposition.

Suppose now one also has $(A,e,e) = (e,A,e) = (e,e,A) = 0$. If, as in the associative case, one takes $x = exe + (ex - exe) + (xe - exe) + (x - ex - xe + exe)$, one sees that $A = A_{11} + A_{10} + A_{01} + A_{00}$ where $A_{ij} = \{x \in A\colon ex = ix, xe = jx\}$. This further decomposition of A is referred to as the Peirce decomposition.

Let A be a generalized alternative algebra II over a field F of characteristic $\neq 2$. When A contains an idempotent e, we will make use of the following results established by Kleinfeld in [5]:

(i) $I = (A,e,e)$ is an ideal of A such that $I^2 = 0$.

(ii) If A permits a Peirce decomposition, then for $i, j, k, t = 0$ or 1 we have $A_{ij}A_{kt} = 0$, when $j \neq k$, except for $A_{01}A_{01} \subseteq A_{10}$ and $A_{10}A_{10} \subseteq A_{01}$. Also $A_{ij}A_{jk} \subseteq A_{ik}$.

Lemma 2.3. *Let A be a generalized alternative algebra* II *over a field F. If B is an ideal of A, then $AB^2 + B^2 = B^2A + B^2$ and B^3 are also ideals of A.*

Proof. Throughout we assume a, $a_i \in A$ and $b_i \in B$ for $i = 1, 2, 3$. Using (2.3) and the fact that B is an ideal of A, we first observe that $(b_1b_2)a = (b_1,b_2,a) + b_1(b_2a) = -(a,b_2,b_1) + b_1(b_2a) = -(ab_2)b_1 + a(b_2b_1) + b_1(b_2a)$ implies $B^2A \subseteq AB^2 + B^2$. Analogously one has $AB^2 \subseteq B^2A + B^2$, and so $AB^2 + B^2 = B^2A + B^2$.

Now from (2.4), $(b_1,b_2a_1,a_2) - (b_1,b_2,a_2a_1) + (b_1,b_2,a_1)a_2 - (b_1,a_1,a_2)b_2 = 0$, we obtain $(B^2A)A \subseteq B^2A + B^2$; whence we have $(B^2A + B^2)A \subseteq B^2A + B^2$. Next (2.4) and (2.3) together give $(a_1,a_2b_1,b_2) - (a_1,a_2,b_2b_1) + (a_1,a_2,b_1)b_2 + (b_2,b_1,a_1)a_2 = 0$. Since we have just shown $(B^2A)A \subseteq B^2A + B^2 = AB^2 + B^2$, we have $A(AB^2) \subseteq AB^2 + B^2$; whence $A(AB^2 + B^2) \subseteq AB^2 + B^2$. Thus $AB^2 + B^2 = B^2A + B^2$ is an ideal of A.

To show B^3 an ideal of A, one needs to show $[b_1(b_2b_3)]a$, $[(b_1b_2)b_3]a$, $a[b_1(b_2b_3)]$, $a[(b_1b_2)b_3] \in B^3$. From (2.4), $(b_1,b_2a,b_3) - (b_1,b_2,b_3a) + (b_1,b_2,a)b_3 - (b_1,a,b_3)b_2 = 0$, we first obtain $[(b_1b_2)a]b_3 \in B^3$ or $(B^2A)B \subseteq B^3$. This and (2.3) then give

$$[(b_1b_2)a]b_3 = [(b_1,b_2,a) + b_1(b_2a)]b_3 = [-(a,b_1,b_2) + b_1(b_2a)]b_3$$

or $(AB^2)B \subseteq B^3$. Similarly $[(b_1b_2)a]b_3 = (b_1b_2,a,b_3) + (b_1b_2)(ab_3) = -(b_3,a,b_1b_2) + (b_1b_2)(ab_3)$ implies $B(AB^2) \subseteq B^3$, which with (2.3) in turn gives

$$b_3[a(b_1b_2)] = b_3[-(a,b_1,b_2) + (ab_1)b_2] = b_3[(b_2,b_1,a) + (ab_1)b_2]$$

or $B(B^2A) \subseteq B^3$.

We are now ready to show B^3 an ideal of A. Since we have just verified that $B(B^2A)$ and $(B^2A)B$ are contained in B^3, from (2.1), $(b_1b_2,b_3,a) + (b_1,b_2,[b_3,a]) = b_1(b_2,b_3,a) + (b_1,b_3,a)b_2$, we have $[(b_1b_2)b_3]a \in B^3$. This and (2.3) then give

$[(b_2b_3)b_1]a = (b_2b_3, b_1, a) + (b_2b_3)(b_1a) = -(a, b_1, b_2b_3) + (b_2b_3)(b_1a)$ or that $a[b_1(b_2b_3)] \in B^3$. Next, since we have now shown $(AB^2)B$ and $A(BB^2)$ to be contained in B^3, from (2.4) we obtain $(a, b_1b_2, b_3) - (a, b_1, b_3b_2) + (a, b_1, b_2)b_3 - (a, b_2, b_3)b_1 = 0$ or $a[(b_1b_2)b_3] \in B^3$. Finally, this and (2.3) yield $a[(b_2b_3)b_1] = -(a, b_2b_3, b_1) + [a(b_2b_3)]b_1 = (b_1, b_2b_3, a) + [a(b_2b_3)]b_1$, whence $[b_1(b_2b_3)]a \in B^3$. This completes the proof of the lemma.

Now, as in the case for standard algebras in [14], let B be any ideal in A, a generalized alternative algebra II. We define $B^{\langle i\rangle}$ inductively by $B^{\langle 0\rangle} = B$, $B^{\langle i+1\rangle} = A(B^{\langle i\rangle})^2 + (B^{\langle i\rangle})^2$. By Lemma 2.3 this gives a descending chain $B^{\langle 0\rangle} \supseteq B^{\langle 1\rangle} \supseteq \cdots \supseteq B^{\langle k\rangle} \supseteq \cdots$ of ideals of A which we call a Penico sequence. We shall call B Penico solvable in case there is some integer $k > 0$ for which $B^{\langle k\rangle} = 0$.

Lemma 2.4. *Let A be a generalized alternative algebra* II *over a field F. An ideal B of A is Penico solvable if and only if B is solvable.*

Proof. If B is Penico solvable, then B is clearly solvable since $B^{\langle i\rangle} \supseteq B^{(i)}$. On the other hand, suppose for any ideal B of A one has $B^{\langle 2\rangle} \subseteq B^{(1)}$. Then, as in the proof of Theorem 3 in [14], induction shows $B^{\langle 2k\rangle} \subseteq B^{(k)}$, since $B^{\langle 2(k+1)\rangle} = (B^{\langle 2k\rangle})^{\langle 2\rangle} \subseteq (B^{\langle 2k\rangle})^{(1)} \subseteq (B^{(k)})^{(1)} = B^{(k+1)}$. Hence, if B is solvable, then $B^{\langle 2k\rangle} \subseteq B^{(k)} = 0$ for some k, that is B is Penico solvable. Now by definition $B^{\langle 2\rangle} = A(AB^2 + B^2)^2 + (AB^2 + B^2)^2$. Since B^3 is an ideal of A, to show $B^{\langle 2\rangle} \subseteq B^3$ it suffices to verify $(AB^2 + B^2)^2 = (AB^2)(AB^2) + (AB^2)B^2 + B^2(AB^2) + B^2B^2 \subseteq B^3$. But, since B an ideal of A implies that AB^2 and B^2 are contained in B, one has $(AB^2)B^2$, $B^2(AB^2)$, and B^2B^2 contained in B^3. Furthermore, since it has been demonstrated in the proof of Lemma 2.3 above that $(AB^2)B \subseteq B^3$, one has $(AB^2)(AB^2) \subseteq (AB^2)B \subseteq B^3$. Thus for any ideal B of A we have $B^{\langle 2\rangle} \subseteq B^3 \subseteq B^2 = B^{(1)}$, and the proof of the lemma is now complete.

Lemma 2.5. *Let A be a generalized alternative algebra* II *over a field F of characteristic $\neq 2$. If A contains an idempotent e, then the ideal $I = (A, e, e)$ satisfies $[A, I] = A_{1/2}I = (A_{1/2})^2I = 0$.*

Proof. As observed in (i), Kleinfeld has shown I to be an ideal of A such that $I^2 = 0$. We also make use of the following observations. From (2.3) it follows that $(A, e, e) = I = (e, e, A)$. Thus (2.7) gives $(e, e, x) = (e, e, ex) + (e, e, xe) = e(e, e, x) + (e, e, x)e$ or $I \subseteq A_{1/2}$. In particular, since (2.7) implies $e(e, e, x) = (e, e, x)e$, we have

(2.v) $ek = \frac{1}{2}k = ke$ for $k \in I$.

Next let $(e, y, z) = a_1 + a_{1/2} + a_0$ and $(e, e, [y, z]) = b_{1/2}$ where $a_i, b_i \in A_i$ for $i = 0, \frac{1}{2}, 1$. Then (2.1) yields $(e, y, z) + (e, e, [y, z]) = e(e, y, z) + (e, y, z)e$ or $a_1 + a_{1/2} + a_0 + b_{1/2} = a_1 + ea_{1/2} + a_1 + a_{1/2}e$, whence $a_1 = a_0 = b_{1/2} = 0$ or

(2.w) $(e, e, [y, z]) = 0$,

(2.x) $(e, y, z) \in A_{1/2}$ for $y, z \in A$.

Since from [2] we know $y, z \in A_{1/2}$ implies $y \circ z \in A_1 + A_0$, we also have $0 = (e, e, yz + zy) + (e, e, yz - zy) = 2(e, e, yz)$ or

(2.y) $(e,e,yz) = 0$ for $y, z \in A_{1/2}$.

Suppose now we are given $x \in A$. Let $x = x_1 + x_{1/2} + x_0$ where $x_i \in A_i$ for $i = 0, \frac{1}{2}, 1$. Then using (2.5), (2.3), (2.v), and (2.w) one has for $i = 0, 1$ and $k \in I$ that $x_i k = x_i(e,e,4k) + (e,e,x_i)(4k) = 4(e,e,x_i k) = 4(e,e,kx_i) = (4k)(e,e,x_i) + (e,e,4k)x_i = kx_i$. Also using (2.y) and the fact that $I^2 = 0$, one has in addition that $0 = (e,e,x_{1/2}k) = x_{1/2}(e,e,k) + (e,e,x_{1/2})k = x_{1/2}(e,e,k) = \frac{1}{4}x_{1/2}k$ as well as $0 = (e,e,kx_{1/2}) = k(e,e,x_{1/2}) + (e,e,k)x_{1/2} = (e,e,k)x_{1/2} = \frac{1}{4}kx_{1/2}$. Thus $[A,I] = 0$ and, in particular, $A_{1/2}I = 0$.

Next let $x, y \in A_{1/2}$ and $k \in I$. Then (2.1) gives $(xy,e,k) + (x,y,[e,k]) = x(y,e,k) + (x,e,k)y$. But $[A,I] = 0$ implies $(x,y,[e,k]) = 0$, while I an ideal of A with $A_{1/2}I = 0$ implies $x(y,e,k) = 0 = (x,e,k)y$. Hence $(xy,e,k) = 0$. Let $xy = a_1 + a_{1/2} + a_0$ where $a_i \in A_i$ for $i = 0, \frac{1}{2}, 1$. Then $0 = (xy,e,k) = [(xy)e]k - \frac{1}{2}(xy)k = (a_1 + a_{1/2}e)k - \frac{1}{2}a_1 k - \frac{1}{2}a_0 k = a_1 k - \frac{1}{2}a_1 k - \frac{1}{2}a_0 k$, using the fact from [7] that for noncommutative Jordan algebras $A_{1/2}A_i, A_iA_{1/2} \subseteq A_{1/2}$ for $i = 0, 1$. Thus we have shown

(2.z) $(xy)_1 k = (xy)_0 k$ for $x, y \in A_{1/2}$ and $k \in I$.

Continuing as above we have $(e,x,y) = (ex)y - e(xy) = (ex)y - a_1 - ea_{1/2}$. Since, by (2.x), $(e,x,y) \in A_{1/2}$, this gives $[(ex)y]_1 = a_1$ and $[(ex)y]_0 = 0$. Then $a_1 + a_{1/2} + a_0 = xy = (ex)y + (xe)y$ implies $[(xe)y]_1 = 0$. Thus $(ex)y \in A_1 + A_{1/2}$ while $(xe)y \in A_{1/2} + A_0$. Now since from [7], as noted above, we know $xe \in A_{1/2}$, (2.z) gives $[(xe)y]_1 k = [(xe)y]_0 k$. But $[(xe)y]_1 = 0$, so $[(xe)y]_1 k = 0 = [(xe)y]_0 k$. Hence $[(xe)y]k = [(xe)y]_1 k + [(xe)y]_{1/2} k + [(xe)y]_0 k = 0$, since $A_{1/2}I = 0$. In similar fashion we have $[(ex)y]k = 0$. But then $x, y \in A_{1/2}$ gives $(xy)k = [(xe + ex)y]k = [(xe)y]k + [(ex)y]k = 0$ or $(A_{1/2})^2 I = 0$. This completes the proof of the lemma.

In proving the next theorem, we will use [8, Lemma 2.1, Theorem 2.1, and Theorem 2.2]. We make note that the exclusion in these results of characteristic 3 is not necessary [16].

Theorem 2.4 (Wedderburn principal theorem). *Let A be a finite-dimensional generalized alternative algebra* II *over a field F of characteristic $\neq 2$, and let N be the nil radical of A. If A/N is separable, then $A = S + N$ (vector space direct sum) where S is a subalgebra of A such that $S \cong A/N$.*

Proof. As in the proof of Theorem 23 on p. 47 of [1], it suffices to prove that A contains a subalgebra $S \cong A/N$. Since our result is true trivially for $N = 0$ or $N = A$, it is certainly true if A has dimension one. We make an induction on the dimension of A and assume the result true for algebras of dimension less than that of A.

From the proof of Theorem 23 on page 47 of [1], it now also follows that one may assume N does not properly contain an ideal of A. Thus we may argue, as in the proof of Theorem 4 of [14], that $N^2 = 0$, for suppose $N^{\langle 1 \rangle} = N$. Since N is solvable by Theorem 2.2, N is Penico solvable by Lemma 2.4. Hence $N = N^{\langle 1 \rangle} = N^{\langle 2 \rangle} = \cdots = N^{\langle k \rangle} = 0$ for some k, and our result is immediate.

Since, by Lemma 2.3, $N^{\langle 1\rangle} \subseteq N$ is an ideal of A, we must then have $0 = N^{\langle 1\rangle} = AN^2 + N^2$, that is $N^2 = 0$.

At this point, an argument analogous to that used for Jordan algebras on page 289 of [3] shows one may also assume the field F to be algebraically closed.

Suppose next that A/N is not a simple algebra. If B is a nodal subalgebra of A/N, then from [11] we know that B has a homomorphic image which is a simple nodal algebra. Since our Theorem 2.3 denies this possibility, we have from Theorem 4 of [11] that A/N semisimple implies $A/N = B_1 + \cdots + B_t$ (algebra direct sum) where each B_i is a simple ideal. Since Theorem 3 of [5] and our Theorem 2.3 imply each B_i is alternative, each B_i must have a unity element. Furthermore, from [7] we know A a noncommutative Jordan algebra implies that A_1 and A_0 are subalgebras of A for any idempotent $e \in A$. From Theorem 2.1 in [8] it now follows that it will suffice to consider the case A/N a simple algebra.

As a final reduction we note, as in the proof of Theorem 2.2 of [8], that if there exists a primitive idempotent e such that our result holds for the ideal H generated by $A_{1/2}$, then it holds for A as well.

Now A/N not nil implies by Proposition 3.3 on p. 32 of [13] that A/N contains a nonzero idempotent e'. Should this be the only nonzero idempotent in A/N, then e' is a unit element for A/N, and Theorem 2.3 implies $A/N = Fe'$. By Lemma 2.1 in [8], e' lifts to an idempotent $e \in A$, and so we have Fe a subalgebra of A such that $Fe \cong A/N$. Hence we may assume that A/N contains a nontrivial idempotent e'. Again e' lifts to an idempotent $e \in A$. In particular, e must be nontrivial and, since A is finite-dimensional, one may assume that e is primitive.

We now let $I = (e,e,A)$. By (i) and (2.3), $(e,e,A) = I = (A,e,e)$ is an ideal of A such that $I^2 = 0$. Since, as earlier observed, we may assume N not to properly contain an ideal of A, we must have either $I = 0$ or $I = N$.

If we suppose first that $I = 0$, then A has a Peirce decomposition relative to e, since, by (2.2), $(e,A,e) = 0$. Let $w_{ij}, x_{ij}, y_{ij}, z_{ij} \in A_{ij}$ for $i,j = 0$ or 1 and consider $H = A_{10}A_{01} + A_{10} + A_{01} + A_{01}A_{10}$. Using the multiplication table described by (ii), it follows that to show H an ideal of A it suffices to show $A_{10}A_{01}$ an ideal of A_{11} and $A_{01}A_{10}$ an ideal of A_{00}. Using (2.3) and the multiplication table described by (ii), one can compute as follows:

$$\begin{aligned}(x_{10}y_{01})z_{11} &= (x_{10},y_{01},z_{11}) + x_{10}(y_{01}z_{11})\\ &= -(z_{11},y_{01},x_{10}) + x_{10}(y_{01}z_{11})\\ &= x_{10}(y_{01}z_{11}) \in A_{10}A_{01}\end{aligned}$$

and

$$\begin{aligned}z_{11}(x_{10}y_{01}) &= -(z_{11},x_{10},y_{01}) + (z_{11}x_{10})y_{01}\\ &= (y_{01},x_{10},z_{11}) + (z_{11}x_{10})y_{01}\\ &= (z_{11}x_{10})y_{01} \in A_{10}A_{01}.\end{aligned}$$

Thus $A_{10}A_{01}$ is an ideal of A_{11}. Similarly one may show $A_{01}A_{10}$ to be an ideal of A_{00}, and hence H is an ideal of A. In particular, H must be the ideal generated by $A_{1/2} = A_{10} + A_{01}$.

Now H a proper ideal implies by the induction hypothesis that our result is valid for H. Thus our final reduction applies, and we may conclude that our result is valid for A itself. On the other hand, should $H = A$ then $A_{11} = A_{10}A_{01}$ and $A_{00} = A_{01}A_{10}$. Using (2.3) and the multiplication table described by (ii), we have

$$\begin{aligned}
[x_{11}(y_{10}z_{01})]w_{11} &= [-(x_{11},y_{10},z_{01}) + (x_{11}y_{10})z_{01}]w_{11} \\
&= [(z_{01},y_{10},x_{11}) + (x_{11}y_{10})z_{01}]w_{11} = [(x_{11}y_{10})z_{01}]w_{11} \\
&= (x_{11}y_{10},z_{01},w_{11}) + (x_{11}y_{10})(z_{01}w_{11}) \\
&= -(w_{11},z_{01},x_{11}y_{10}) + (x_{11}y_{10})(z_{01}w_{11}) = (x_{11}y_{10})(z_{01}w_{11}) \\
&= (x_{11},y_{10},z_{01}w_{11}) + x_{11}[y_{10}(z_{01}w_{11})] \\
&= -(z_{01}w_{11},y_{10},x_{11}) + x_{11}[y_{10}(z_{01}w_{11})] = x_{11}[y_{10}(z_{01}w_{11})] \\
&= x_{11}[-(y_{10},z_{01},w_{11}) + (y_{10}z_{01})w_{11}] \\
&= x_{11}[(w_{11},z_{01},y_{10}) + (y_{10}z_{01})w_{11}] \\
&= x_{11}[(y_{10}z_{01})w_{11}].
\end{aligned}$$

Since $A_{11} = A_{10}A_{01}$, these calculations show A_{11} to be associative. Similarly one may show A_{00} to be associative. If one then joins the calculations on p. 337 of [5], one may conclude that A itself is an alternative algebra. But then from [9] our result is known to be valid for A, and the induction is complete.

Consider now the second alternative, namely $I = N$, and take $k = (e,e,x) \neq 0$. We recall that, since A is noncommutative Jordan, one has from [7] that $A_{1/2}A_i, A_iA_{1/2} \subseteq A_{1/2}$ for $i = 0, 1$. In particular, this says that $N = I = (e,e,A) \subseteq A_{1/2}$. Let H be the ideal in A generated by $A_{1/2}$, then $H = A_{1/2} + (A_{1/2})^2$. To see this, let $x_i, y_i, z_i \in A_i$ for $i = 0, \frac{1}{2}, 1$. Then for $i = 0, 1$ we have $(x_{1/2}y_{1/2})z_i = (x_{1/2},y_{1/2},z_i) + x_{1/2}(y_{1/2}z_i) = (x_{1/2},y_{1/2} + z_i,y_{1/2} + z_i) - (x_{1/2},y_{1/2},y_{1/2}) - (x_{1/2},z_i,z_i) - (x_{1/2},z_i,y_{1/2}) + x_{1/2}(y_{1/2}z_i)$ is in $N + (A_{1/2})^2 \subseteq A_{1/2} + (A_{1/2})^2$, since A/N simple implies as before that A/N is alternative or that $(a,b,b) \in N$ for all a, $b \in A$. Similarly one has $z_i(x_{1/2}y_{1/2}) \in A_{1/2} + (A_{1/2})^2$ for $i = 0, 1$. Since the cases for $i = \frac{1}{2}$ are immediate if one writes $x_{1/2}y_{1/2} = a_1 + a_{1/2} + a_0$ where $a_i \in A_i$, we have $H = A_{1/2} + (A_{1/2})^2$ as claimed. Now, by Lemma 2.5, $Hk = 0$, while by (2.v) of Lemma 2.5 $ek = \frac{1}{2}k \neq 0$. Thus, since $e \notin H$, H is a proper ideal of A. Our final reduction now applies to complete the induction and the proof of the theorem.

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